Q. State and prove Cayley-Hamilton Theorem.

Sol.ⁿ **Statement:-** Every square matrix satisfies its characteristic equation.

Let, $A=(a_{ij})_{n\times n}$ be any square matrix of order n.

The characteristic equation of A is -

$$|A - xI_n| = 0$$

$$\Rightarrow \begin{vmatrix} a_{11}-x, & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-x & a_{23} & \cdots & a_{2n} \\ - & - & - & - & - \\ - & - & - & - & - \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-x \end{vmatrix} = 0$$

$$\Rightarrow a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0 \text{ (say)} \rightarrow ①$$

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We are to show that,

$$a_0 I_n + a_1 A + a_2 A^2 + \cdots + a_n A^n = 0$$

We now consider adj.$(A - xI_n)$.

Since, each element of adj.$(A - xI_n)$ is a polynomial in x of degree atmost $(n-1)$. Therefore adj.$(A-xI_n)$ can be expressed as a matrix polynomial as –

$$\text{adj}\cdot(A - xI_n) = B_0 + B_1 x + B_2 x^2 + \cdots + B_{n-1} x^{n-1} \longrightarrow (2)$$

where, $B_0, B_1, \cdots B_{n-1}$ are square matrix of order n.

But we know that

$$\text{adj}\cdot(A - xI_n)\cdot(A - xI_n) = |A - xI_n|\, I_n \qquad [\because A(\text{adj}A) = (\text{adj}A)A = |A|\cdot I_n]$$

$$\Rightarrow (B_0 + B_1 x + B_2 x^2 + \cdots + B_{n-1} x^{n-1})(A - xI_n) = (a_0 + a_1 x + a_2 x^2 \cdots + a_n x^n) I_n$$

$$\Rightarrow B_0 A - x B_0 I_n + B_1 A x - B_1 I_n x^2 + B_2 A x^2 - B_2 I_n x^3 + \cdots + B_{n-1} A x^{n-1} - B_{n-1} I_n x^n$$
$$= a_0 I_n + (a_1 I_n) x + (a_2 I_n) x^2 \cdots + (a_n I_n) x^n$$

$$\Rightarrow B_0 A + (B_1 A - B_0 I_n) x + (B_2 A - B_1 I_n) x^2 + \cdots + (B_{n-1}A - B_{n-2} I_n) x^{n-1} - B_{n-1} I_n x^n$$
$$= a_0 I_n + (a_1 I_n) x + (a_2 I_n) x^2 + \cdots + (a_{n-1} I_n) x^{n-1} + (a_n I_n) x^n$$

Equating the co-eff^{nt} of like power of x on both sides we get,

$$a_0 I_n = B_0 A$$
$$a_1 I_n = B_1 A - B_0 I_n$$
$$a_2 I_n = B_2 A - B_1 I_n$$
$$\cdots\cdots\cdots$$
$$\cdots\cdots\cdots$$
$$a_{n-1} I_n = B_{n-1} A - B_{n-2} I_n$$
$$a_n I_n = -B_{n-1} I_n$$

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Now multiplying the above equation by $I_n, A, A^2, \cdots, A^n$ resp^{ly} and then adding we get,

$$a_0 I_n + a_1 A + a_2 A^2 + \cdots + a_n A^n = 0$$

This shows that 'A' satisfies its characteristic equation.

Note:- $a_0 I_n + a_1 A + a_2 A^2 + \cdots + a_{n-1} A^{n-1} + a_n A^n = 0$

Multiplying both sides by A^{-1}, we get

$$\Rightarrow a_0 A^{-1} + a_1 I_n + a_2 A + \cdots + a_{n-1} A^{n-2} + a_n A^{n-1} = 0$$

This will gives us the way to determine the inverse of a matrix with the help of Cayley-Hamilton Theorem.

Dt. 20.10.08

Q. Varify Cayley - Hamilton Theorem for the following matrix and hence find A^{-1}!

08 (i) $A=\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 05 (ii) $A=\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$ 06 (iii) $A=\begin{bmatrix} 1 & 3 & 0 \\ -2 & 2 & -1 \\ 4 & 0 & -2 \end{bmatrix}$

06 (iv) $A=\begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 3 & 2 & -3 \end{bmatrix}$

$A^3-A^2+2A+28=0$

Sol.^{n} (ii) Given that, $A=\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

The characteristic equation of the matrix A is,

$$|A-xI|=0$$

$$\Rightarrow \begin{vmatrix} 3-x & 1 \\ -1 & 2-x \end{vmatrix}=0$$

$$\Rightarrow (3-x)(2-x)+1=0$$

$$\Rightarrow 6-5x+x^2+1=0$$

$$\Rightarrow x^2-5x+7=0$$

We are to show that,

$$A^2-5A+7I=0$$

Now, $A^2=A.A=\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

$$=\begin{pmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{pmatrix}$$

$$=\begin{pmatrix} 8 & 5 \\ -5 & 3 \end{pmatrix}$$

Now, $A^2-5A+7I=\begin{pmatrix} 8 & 5 \\ -5 & 3 \end{pmatrix}-5\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}+7\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$=\begin{pmatrix} 8 & 5 \\ -5 & 3 \end{pmatrix}-\begin{pmatrix} 15 & 5 \\ -5 & +10 \end{pmatrix}+\begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}$$

$$=\begin{pmatrix} 8-15+7 & 5-5+0 \\ -5-5+10 & 3-10+7 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}=0$$

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