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Q:→ Show that $A = \{2n : n \in \mathbb{Z}\}$ with addition and multiplication as defined in $\mathbb{Z}$, the set of integers is a ring, while $B = \{2n+1 : n \in \mathbb{Z}\}$ is not.

Soln:→ Here, $A = \{2n : n \in \mathbb{Z}\}$

$= \{0, \pm 2, \pm 4, \pm 6, \cdots\}$

= The set of even integer.

We now show that A is a ring.

R_1> (i) Since sum of two even integer is again an even integer.

$\therefore x+y \in A, \forall x, y \in A.$

(ii) Again the set of even integer satisfies associative law w.r. to addition.

$\therefore x+(y+z) = (x+y)+z, \forall x, y, z \in A$

(iii) clearly $0 \in A$, which is the additive identity of A.

(iv) For any $x \in A$, we have $-x \in A$ and

$x+(-x) = (-x)+x = 0.$

(v) Again the set of even integer satisfies commutative law w.r. to addition.

$\therefore x+y = y+x, \forall x, y \in A.$

Hence A is an additive abelian group.

R_2> The set of even integer satisfies associative law w.r. to multiplication.

$\therefore x(yz) = (xy)z, \forall x, y, z \in A.$

R_3> Moreover, we have,

$x(y+z) = xy + xz$

$\& (y+z)x = yx + zx, \forall x, y, z \in A$

Hence A is a ring.

2nd part:-

Again, $B = \{2n+1 : n \in \mathbb{Z}\}$

$= \{\pm 1, \pm 3, \pm 5, \pm 7, \cdots\}$

Clearly B is not ring. For,

$1, 3 \in B \Rightarrow 1+3 = 4 \notin B$. #

06. Q:→ The operation (+) and (·) are defined on the set $\mathbb{Z}$ of integers by

$a \oplus b = a+b-1$.

$a \odot b = a+b-ab, \forall a, b \in \mathbb{Z}$

Show that $(\mathbb{Z}, \oplus, \odot)$ is a commutative ring with unity.

Soln:→ Here, the operation $\oplus$ and $\odot$ are defined on the set $\mathbb{Z}$ of integers by

$a \oplus b = a+b-1$

$a \odot b = a+b-ab, \forall a, b \in \mathbb{Z}$.

R_1) (i) ~~For any~~ Let $a, b \in \mathbb{Z}$

$\Rightarrow a+b \in \mathbb{Z}$

$\Rightarrow a+b-1 \in \mathbb{Z}$

$\Rightarrow a \oplus b \in \mathbb{Z}, \forall a, b \in \mathbb{Z}$

(ii) Let $a, b, c \in \mathbb{Z}$.

~~a+(b+c~~

$a \oplus (b \oplus c) = a \oplus (b+c-1) = a + (b+c-1) - 1$

$= a+b+c-2$.

Again, $(a \oplus b) \oplus c = (a+b-1) \oplus c$

$\therefore a \oplus (b \oplus c) = (a \oplus b) \oplus c, \forall a,b,c \in \mathbb{Z}.$

(iii) Let, 'e' be the additive identity of $\mathbb{Z}$.

Then, $a \oplus e = a$

$$\cancel{a} + e - 1 = \cancel{a}$$

$$\Rightarrow e = 1.$$

This shows that 1 is the additive identity of $\mathbb{Z}$.

(iv) Let $a \in \mathbb{Z}$ and a_1 be the additive inverse of a.

$$\therefore a \oplus a_1 = e$$

$$\Rightarrow a + a_1 - 1 = 1$$

$$\Rightarrow a + a_1 = 2$$

$$\Rightarrow a_1 = 2 - a$$

$\therefore 2-a$ is the additive inverse of 'a'.

(v) For any $a,b \in \mathbb{Z}$, we have,

$$b \oplus a = b + a - 1$$
$$= a + b - 1$$
$$= a \oplus b$$

$\therefore a \oplus b = b \oplus a, \forall a,b \in \mathbb{Z}.$

Hence $(\mathbb{Z}, \oplus)$ is an abelian group.

R_2> For any $a,b,c \in \mathbb{Z}$, we have,

$$a \odot (b \odot c) = a \odot (b + c - bc)$$
$$= a + (b + c - bc) - a(b + c - bc)$$
$$= a + b + c - bc - ab - ac + abc$$

Again, $(a \odot b) \cdot c =$