(ii) We have,

$x+x=0$

Now, $x+y=0$

$\Rightarrow x+y=x+x$

$\Rightarrow y=x$ (by left cancellation law).

$\Rightarrow x=y$

Proved

(iii) For any $x,y \in R$, we have –

$x+y \in R$

$\Rightarrow (x+y)^2 = x+y$ [using (i)]

$\Rightarrow (x+y)(x+y) = x+y$

$\Rightarrow x(x+y)+y(x+y) = (x+y)$

$\Rightarrow x^2+xy+yx+y^2 = (x+y)$

$\Rightarrow x+xy+yx+y = (x+y)$ [using (i)]

$\Rightarrow (x+y)+(xy+yx) = (x+y)+0$

$\Rightarrow xy+yx=0$ (by left cancellation law)

$\Rightarrow xy=-yx$ ——(2)

$\Rightarrow x(xy)=x(-yx)$

$\Rightarrow (xx)y = -(xyx)$ [$\because a(-b)=-(ab)$]

$\Rightarrow x^2y=-(xyx)$

$\Rightarrow xy=-(xyx)$ ——(3)

Again, from (2) we have,

$xy=-yx$

$\Rightarrow xyx=(-yx)x$

$\Rightarrow xyx=(-y)(xx)$

$\Rightarrow xyx=(-y)x^2$

$\Rightarrow xyx=(-y)x$

$\Rightarrow xyx=-(yx)$

$\Rightarrow -(xyx)=yx$ ——(4)

from (3) & (4) we get,

$xy=yx$

Proved

Alt. proof of (iii) :→ (with (ii) proof)

For any $x, y \in R$, we have,

$x+y \in R$

$\Rightarrow (x+y)^2 = x+y \quad [\because x^2 = x, \forall x \in R]$

$\Rightarrow (x+y)(x+y) = x+y$

$\Rightarrow x(x+y) + y(x+y) = x+y$

$\Rightarrow x^2 + xy + yx + y^2 = x+y$

$\Rightarrow x + xy + yx + y = x+y$

$\Rightarrow (x+y) + (xy+yx) = (x+y) + 0$

$\Rightarrow xy + yx = 0$ (by left cancellation law).

$\Rightarrow xy = yx \quad [\because x+y = 0 \Rightarrow x = y]$.

Proved

08 Theorem :→ A ring $R$ is without zero divisor if and only if cancellation law hold good in $R$.

Proof :→ We are to show that,

$R$ is without zero divisor $\Leftrightarrow$ Cancellation law hold good in $R$.

First, let $R$ be without zero divisor and let $a \neq 0, b, c \in R$ such that, $ab = ac$

$\therefore ab = ac$

$\Rightarrow ab - ac = 0$

$\Rightarrow a(b-c) = 0$

$\Rightarrow b - c = 0 \quad [\because a \neq 0$ & $R$ is without zero divisor$]$

$\Rightarrow b = c$

This shows that cancellation law hold good in $R$.

Conversely, let cancellation law hold good in $R$

We now show that $R$ is without zero divisor

Let $a, b \in R$ such that $ab = 0$. To show either $a = 0$ or $b = 0$

let $a \neq 0$. To show $b = 0$.

Now, $ab=0$

$\Rightarrow ab = a0$

$\Rightarrow b=0$ (by left cancellation law).

Similarly if $b\neq 0$, Then we can show that $a=0$

$\therefore ab=0 \Rightarrow$ either $a=0$ or $b=0$

This shows that $R$ is without zero divisor.

*** 04.06 Theorem :→ Prove that Every field is an integral domain. Is The converse true? Justify your answer.

Proof :→ Let $R$ be a field. In order to show that $R$ is an integral domain, it is sufficient to show that $R$ is without zero divisor.

let $a,b\in R$ such that $ab=0$. To show either $a=0$ or $b=0$

let $a\neq 0$. To show $b=0$.

Since $a\neq 0\in R$, $a^{-1}\in R$ and $aa^{-1}=a^{-1}a=1$.

Now, $ab=0$

$\Rightarrow a^{-1}(ab) = a^{-1}0$

$\Rightarrow (a^{-1}a)\,b=0$

$\Rightarrow 1\,b=0$

$\Rightarrow b=0$.

Similarly, if $b\neq 0$, Then we can show that $a=0$.

$\therefore ab=0 \Rightarrow$ either $a=0$ or $b=0$, $\forall\, a,b\in R$

This show that $R$ is without zero divisor

Hence, $R$ is an integral domain.

2nd part :→ The converse of The above result may not be true. That means an integral domain may not be a field. For this we cite an example below —

The set of integer $\mathbb{Z}$ is an integral domain but it is not a field. For $2\in\mathbb{Z}$, but $2^{-1}=\frac{1}{2}\notin\mathbb{Z}$.