Linear Algebra

Defination of vector space:- Let. V be a non-empty set and F be a field. If there exists an internal binary operation (+), called vector addition such that:-

V_1 > (V, +) is an abelian group i.e.

(i) $x+y \in V, \forall x, y \in V$

(ii) $x+(y+z)=(x+y)+z, \forall x, y, z \in V$

(iii) $\exists \bar{0} \in V$ such that

$$x+\bar{0}=\bar{0}+x=x, \forall x \in V.$$

(iv) For any $x \in V$, $\exists -x \in V$ such that

$$x+(-x)=(-x)+x=\bar{0}$$

(v) $x+y=y+x, \forall x, y \in V$

Again there exists an external binary operation (·), called scalar multiplication such that

V_2 > $\alpha(x+y)=\alpha.x+\alpha.y, \forall x, y \in V$ & $\forall \alpha \in F$

V_3 > $(\alpha+\beta).x=\alpha.x+\beta.x, \forall x \in V$ & $\forall \alpha, \beta \in F$

V_4 > $(\alpha\beta).x=\alpha(\beta x), \forall x \in V, \forall \alpha, \beta \in F$

V_5 > $1.x=x, \forall x \in V$

where 1 is the multiplicative identity of the field F

Then V is said to be a vector space over F and it is denoted by V(F).

Note:- 1. If V(F) is a vector space then the elements of V are said to be vectors which are generaly denoted by, x, y, z or u, v, w etc., where as the element of the field F are said to be the scatar and are generaly denoted by α, β, γ or a, b, c etc.

2. The additive identity of V is denoted by $\bar{0}$ or θ where as the additive identity of the field F is denoted by 0.

Some Example of vector space

Q. 1. Let $\mathbb{R}^n$ denotes the set consisting of all ordered n tuples of real numbers of the form $x=(x_1,x_2,x_3 \cdots x_n)$. define suitable vector addition and scalar multiplication and show That $\mathbb{R}^n$ is a vector space over the field $\mathbb{R}$.

Sol^n: Let us define vector addition and scalar multiplication by,

i. Vector addition :- $x+y=(x_1,x_2,x_3,\cdots x_n)+(y_1,y_2,y_3\cdots y_n)$

$$=(x_1+y_1, x_2+y_2, x_3+y_3, \ldots x_n+y_n)$$

$$\forall\, x=(x_1,x_2,x_3\cdots x_n)\in \mathbb{R}^n$$

$$y=(y_1,y_2,y_3\cdots y_n)\in \mathbb{R}^n$$

ii. Scalar multiplication :- $\alpha x = \alpha(x_1,x_2,x_3,\cdots x_n)$

$$=(\alpha x_1, \alpha x_2, \alpha x_3, \ldots \alpha x_n)$$

$$\forall \alpha\in\mathbb{R}\ \&\ x=(x_1,x_2,\ldots x_n)\in\mathbb{R}^n.$$

We now show that $\mathbb{R}^n(\mathbb{R})$ is a vector space.

V_1) (i) From the defination it is clear that,

$$x+y\in\mathbb{R}^n,\ \forall\, x,y\in\mathbb{R}^n.$$

(ii) We now have

$$x+(y+z)=(x_1,x_2,x_3,\ldots x_n)+\{(y_1,y_2,y_3\cdots y_n)+(z_1,z_2\ldots z_n)\}$$

$$=(x_1,x_2,x_3,\cdots x_n)+\{(y_1+z_1),(y_2+z_2),\ldots,(y_n+z_n)\}$$

$$=(x_1+(y_1+z_1),\ x_2+(y_2+z_2),\ \ldots\ x_n+(y_n+z_n))$$

$$=((x_1+y_1)+z_1,\ (x_2+y_2)+z_2,\ \ldots,(x_n+y_n)+z_n)$$

$$=(x_1+y_1, x_2+y_2, \ldots x_n+y_n)+(z_1,z_2,\ldots z_n)$$

$$=\{(x_1,x_2,\ldots x_n)+(y_1,y_2,\ldots y_n)\}+z$$

$$=(x+y)+z,\ \forall\, x,y,z\in\mathbb{R}^n.$$

(iii) Clearly $\bar{0}=(0,0,0,\ldots,0)\in\mathbb{R}^n$

We now have –

$$x+\bar{0}=(x_1,x_2,\ldots,x_n)+(0,0,\ldots,0)$$
$$=(x_1+0,\ x_2+0,\ldots,x_n+0)$$
$$=(x_1,x_2,\ldots,x_n)$$
$$=x$$

Similarly we can show that

$$\bar{0}+x=x$$

$\therefore\ x+\bar{0}=\bar{0}+x=x,\ \forall\ x\in\mathbb{R}^n$

This shows that $\bar{0}$ is the additive identity of $\mathbb{R}^n$.

(iv) For any $x=(x_1,x_2,\ldots x_n)\in\mathbb{R}^n$, we have

$$-x=(-x_1,-x_2,\ldots-x_n)\in\mathbb{R}^n$$

$$\therefore x+(-x)=(x_1,x_2,\ldots,x_n)+(-x_1,-x_2,\ldots-x_n)$$
$$=(x_1+(-x_1),\ x_2+(-x_2),\ldots,x_n+(-x_n))$$
$$=(0,0,\ldots,0)$$
$$=\bar{0}$$

$\therefore x+(-x)=\bar{0}$

Similarly we can show that

$$(-x)+x=\bar{0}$$

$$x+(-x)=(-x)+x=\bar{0}$$

This shows that $-x$ is the additive inverse of x.

(v) Again we have –

$$x+y=(x_1,x_2,\ldots,x_n)+(y_1,y_2,\ldots y_n)$$
$$=(x_1+y_1,\ x_2+y_2,\ldots,x_n+y_n)$$
$$=(y_1+x_1,\ y_2+x_2,\ldots,y_n+x_n)$$ [by Commutativity in $\mathbb{R}$]
$$=(y_1,y_2,\ldots,y_n)+(x_1,x_2,\ldots,x_n)$$
$$=y+x$$

$\therefore x+y=y+x,\ \forall\ x,y\in\mathbb{R}^n$

$\therefore$ Hence, $\mathbb{R}^n$ is an additive abelian group

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