$$= \{(a_0 + a_1x + \cdots + a_nx^n) + (b_0 + b_1x + \cdots + b_nx^n)\} + (c_0 + c_1x + \cdots + c_px^p) \quad [\because m = n = p]$$

$$= \{P(x) + Q(x)\} + R(x)$$

$$\therefore P(x) + \{Q(x) + R(x)\} = \{P(x) + Q(x)\} + R(x), \quad \forall P(x), Q(x), R(x) \in F[x]$$

(iii) Clearly $0 = 0 + 0x + 0x^2 + \cdots + 0x^n \in F[x]$

We now have,

$$P(x) + 0 = 0 + P(x), \quad \forall P(x) \in F[x]$$

This shows that '0' is the additive identity of $F[x]$.

(iv) For any $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in F[x]$, we have

$$-P(x) = (-a_0) + (-a_1)x + (-a_2)x^2 + \cdots + (-a_n)x^n \in F[x].$$

We now have –

$$P(x) + (-P(x)) = (a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) + ((-a_0) + (-a_1)x + (-a_2)x^2 + \cdots + (-a_n)x^n)$$

$$= (a_0 - a_0) + (a_1 - a_1)x + (a_2 - a_2)x^2 + \cdots + (a_n - a_n)x^n$$

$$= 0 + 0x + 0x^2 + \cdots + 0x^n$$

$$= 0$$

Slly, we can show that,

$$(-P(x)) + P(x) = 0$$

$$\therefore P(x) + (P(-x)) = (P(-x)) + P(x) = 0$$

This shows that $-P(x)$ is the additive inverse of $P(x)$.

(v) Again we have,

$$P(x) + Q(x) = (a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) + (b_0 + b_1x + b_2x^2 + \cdots + b_nx^n)$$

$$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n \quad \text{if } n = m$$

$$= (b_0 + a_0) + (b_1 + a_1)x + (b_2 + a_2)x^2 + \cdots + (b_n + a_n)x^n$$

$$= (b_0 + b_1x + b_2x^2 + \cdots + b_nx^n) + (a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)$$

$$= Q(x) + P(x)$$

$$\therefore P(x) + Q(x) = Q(x) + P(x), \quad \forall P(x), Q(x) \in F[x]$$

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Hence, $F[x]$ is an additive abelian group.

V_2) For any $\alpha, \beta \in F$, we have –

$$\alpha(P(x)+Q(x)) = \alpha((a_0+a_1x+a_2x^2+\dots+a_nx^n)+(b_0+b_1x+b_2x^2+\dots+b_mx^m))$$
$$= \alpha((a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2+\dots+(a_n+b_m)x^n)$$
if $n=m$
$$= \alpha(a_0+b_0)+\alpha(a_1+b_1)x+\dots+\alpha(a_n+b_m)x^n$$
$$= (\alpha a_0+\alpha b_0)+(\alpha a_1+\alpha b_1)x+\dots+(\alpha a_n+\alpha b_m)x^n$$
$$= (\alpha a_0+(\alpha a_1)x+\dots+(\alpha a_n)x^n)+(\alpha b_0+(\alpha b_1)x+\dots+(\alpha b_m)x^m)$$
$$= \alpha(a_0+a_1x+\dots+a_nx^n)+\alpha(b_0+b_1x+\dots+b_mx^m)$$
$$= \alpha P(x)+\alpha Q(x)$$

$\therefore \alpha(P(x)+Q(x)) = \alpha P(x)+\alpha Q(x), \forall \alpha \in F \,\&\, P(x), Q(x) \in F[x]$

V_3) Again we have,

$$(\alpha+\beta)P(x) = (\alpha+\beta)(a_0+a_1x+\dots+a_nx^n)$$
$$= (\alpha+\beta)a_0+(\alpha+\beta)a_1x+\dots+(\alpha+\beta)a_nx^n$$
$$= (\alpha a_0+\beta a_0)+(\alpha a_1+\beta a_1)x+\dots+(\alpha a_n+\beta a_n)x^n$$
$$= (\alpha a_0+\alpha a_1x+\dots+\alpha a_nx^n)+(\beta a_0+\beta a_1x+\dots+\beta a_nx^n)$$
$$= \alpha(a_0+a_1x+\dots+a_nx^n)+\beta(a_0+a_1x+\dots+a_nx^n)$$
$$= \alpha P(x)+\beta P(x)$$

$\therefore (\alpha+\beta)P(x) = \alpha P(x)+\beta P(x) \;\forall \alpha,\beta \in F \,\&\, P(x) \in F[x]$

V_4) We have,

$$(\alpha\beta)P(x) = (\alpha\beta)(a_0+a_1x+a_2x^2+\dots+a_nx^n)$$
$$= ((\alpha\beta)a_0+(\alpha\beta)a_1x+(\alpha\beta)a_2x^2+\dots+(\alpha\beta)a_nx^n)$$
$$= (\alpha(\beta a_0)+\alpha(\beta a_1x)+\alpha(\beta a_2x^2)+\dots+\alpha(\beta a_nx^n))$$
$$= \alpha(\beta a_0+\beta a_1x+\beta a_2x^2+\dots+\beta a_nx^n)$$
$$= \alpha(\beta(a_0+a_1x+a_2x^2+\dots+a_nx^n))$$
$$= \alpha(\beta P(x))$$

$\therefore (\alpha\beta)P(x) = \alpha(\beta P(x)) \;\forall \alpha,\beta \in F \,\&\, P(x) \in F[x]$

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V_5) We have,

$$1 . P(x) = 1 (a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)$$
$$= (1.a_0 + 1.a_1x + 1.a_2x^2 + \cdots + 1.a_nx^n)$$
$$= (a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)$$
$$= P(x)$$

$\therefore 1. P(x) = P(x), \forall P(x) \in F[x]$

where 1 is the multiplicative identity of the field F.

$\therefore$ F[x] is a vector space over the field F. //

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Theorem:- Prove that in a vector space X(F),

(i) $\alpha.\bar{0} = \bar{0}$

(ii) $0.\bar{x} = \bar{0}$

(iii) $(-\alpha).\bar{x} = \alpha.(-\bar{x}) = -(\alpha.\bar{x})$

(iv) $(-1).\bar{x} = -\bar{x}$

(v) If $\alpha\bar{x} = \bar{0}$, then either $\alpha = 0$ or $\bar{x} = \bar{0}$

where $\alpha \in F, x \in X$ and $\bar{0}, 0$ are the zero of X & F respectively.

Sol^n Given that X(F) is a vector space.

(i) $\because \bar{0} \in X$

$\therefore \bar{0} + \bar{0} = \bar{0}$

$\Rightarrow \alpha.(\bar{0} + \bar{0}) = \alpha.\bar{0}$

$\Rightarrow \alpha.\bar{0} + \alpha.\bar{0} = \alpha.\bar{0}$

$\Rightarrow \alpha.\bar{0} + \alpha.\bar{0} = \alpha.\bar{0} + \bar{0}$

$\Rightarrow \alpha.\bar{0} = \bar{0}$ [by left cancellation law]

$\therefore \alpha.\bar{0} = \bar{0}, \forall \alpha \in F$ & $\bar{0} \in X$.

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