$x = \frac{1}{x} = x^{-1}$

0 has no inverse

$\frac{1}{0} = \infty$.

$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots$

$Q_0 = \{$

$R_0 =$ non zero

## Group

**Definition :-** A non-empty set $G$ together with a binary operation $*$ is said to be a group if it satisfies the following conditions.

(I) $x * y \in G, \forall x, y \in G$ (Closure property).

(II) $x * (y * z) = (x * y) * z, \forall x, y, z \in G$ (associative property).

(III) There exists an element $e \in G$ s.t

$$x * e = e * x = x, \forall x \in G \text{ (Existance of identity)}.$$

(IV) For any $x \in G$, $\exists$ an element $x_1 \in G$ s.t

$$x * x_1 = x_1 * x = e \text{ [existance of inverse]}.$$

**Note:-** (I) The element $e$ as defined above is said to be the identity element of the group $G$. and $x_1$ is said to be the inverse of $x$ and in this case we write, $x_1 = x^{-1}$.

$$\therefore x * x^{-1} = x^{-1} * x = e.$$

II. If there is no doubt about the binary operation then the above four condition can be written as —

(I) $xy \in G, \forall x, y \in G$

(II) $x(yz) = (xy)z, \forall x, y, z \in G$

(III) $x \cdot e = ex = x, \forall x \in G$

(IV) $x x_1 = x_1 x = e$

or $x x^{-1} = x^{-1} x = e$.

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III. A set $G$ is a group w.r.to addition $(+)$ as binary operation if

(i) $x+y \in G$

(ii) $x+(y+z)=(x+y)+z,\ \forall x,y,z \in G.$

(iii) $x+e=e+x=x,\ \forall x \in G.$

In this case $e$ is said to be the additive identity of $G$.

(iv) For any $x \in G$, $\exists\ x_1 \in G$. s.t.

$$x+x_1=x_1+x=e$$

In this case $x_1$ is said to be the additive inverse of $x$. and we write it as $x_1=-x$.

IV. A set $G$ is a group w.r.to multiplication $(\cdot)$ as binary operation if

(i) $x.y \in G,\ \forall x,y \in G$

~~(ii) x.(x~~

(ii) $x.(y.z)=(x.y).z,\ \forall x,y,z \in G$

(iii) $xe=ex=x,\ \forall x \in G$

In this case '$e$' is said to be the multiplicative identity of $G$.

(iv) For any $x \in G$, $\exists\ x_1 \in G$. s.t

$$x.x_1=x_1.x=e$$

In this case $x_1$ is said to be the multiplicative inverse of $x$ and we write it as $x_1=x^{-1}$.

V. A group $G$ is said to be a commutative group or abelian group if it satisfies commutative law. i.e,

$$x*y=y*x,\ \forall x,y \in G$$

or $x.y=y.x$ (w.r.to multiplication)

or $x+y=y+x$ (w.r.to addition).

Date 22/4/09

Q. 07 Q:→ Prove that the set of matrices

$$A_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

where $\alpha$ is a real number, forms a group under matrix multiplication.

Soln: (i) Let $G = \left\{ A_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$

We now show that $G$ is a group w.r.to matrix multiplication. Let $A_\alpha, A_\beta \in G$. Then,

$$A_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \;\&\; A_\beta = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}.$$

Now,

$$A_\alpha . A_\beta = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{pmatrix}.$$

$$= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}.$$

$$= A_{\alpha+\beta} \in G.$$

$\therefore A_\alpha . A_\beta \in G, \forall A_\alpha, A_\beta \in G.$

(ii) We know that matrix multiplication satisfies associative law.

$\therefore A_\alpha\cdot(A_\beta\cdot A_\gamma) = (A_\alpha\cdot A_\beta)A_\gamma, \forall A_\alpha, A_\beta, A_\gamma \in G.$

(iii) Clearly $A_0 = \begin{pmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in G$, which is the identity element of $G$.