(iv) For any $A_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \in G$, we have,

$$|A_\alpha| = \begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix}$$

$$= \cos^2\alpha + \sin^2\alpha$$

$$= 1.$$

$$\neq 0.$$

$\because |A_\alpha| \neq 0$, Therefore A_α^{-1} exists.

Thus every elements of G has its multiplicative inverse.

Hence G is a group.

{G.U.} '04

Q:- Show that the set

$$G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R}, ad - bc \neq 0 \right\}$$

is a group w.r.to matrix multiplication. Is this group abelian? Justify your answer. 3+1+1.

Sol^n:- Given that,

$$G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R}, ad - bc \neq 0 \right\}$$

We now show that G is a group w.r.to matrix multiplication.

(i) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ & $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ be any two elements of G. Then,

$\cancel{ab}$ $ad - bc \neq 0$

& $ps - qr \neq 0$.

$\therefore AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix}$

$$= \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix}.$$

Now, $(ap+br)(cq+ds)-(cp+dr)(aq+bs)$

$= \cancel{acpq} + apds + bcqr + \cancel{bdrs} - \cancel{acpq} - bcps - adqr - \cancel{bdrs}$

$= ad(ps-qr) - bc(ps-qr)$

$= (ps-qr)(ad-bc)$.

$\neq 0$.

$\therefore AB \in G, \forall A, B \in G$.

(ii) Since matrix multiplication satisfies associative law.

$\therefore A(BC) = (AB).C \quad , \forall A, B, C \in G$.

(iii) clearly $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in G$. which is the identity element of G.

(iv) for any $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$, we have -

$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$= ad - bc$
$\neq 0$

A^{-1} exists.

Thus every element of G has its multiplicative inverse.

Hence G is a group.

2nd part:- The group G is not abelian, for this we cite an example below:-

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

Now, $AB =$

$BA =$

$\therefore AB \neq BA.$

Date 04/12/09 04

Q:→ Prove that the set G of all odd integers forms a group w.r.t. to the composition $*$ defined by $a*b=a+b-1$, for all $a, b \in G$.

Soln:→ Given that –

$G =$ The set of odd integers.

$= \{\pm 1, \pm 3, \pm 5, \pm 7, ----\}.$

Define a binary operation $*$ on G by,

$a*b=a+b-1, \ \forall a, b \in G.$

We are to show that $(G,*)$ is a group.

(i) Let $a, b \in G$

$\Rightarrow a, b$ are odd integer

$\Rightarrow a+b$ is an even integer

$\Rightarrow a+b-1$ is an odd integer.

$\Rightarrow a+b-1 \in G$

$\Rightarrow a*b \in G, \ \forall a, b, \in G.$

(ii) Let $a, b, c \in G$

We now have,

$a*(b*c) = a*(b+c-1)$

$= a+(b+c-1)-1$

$= a+b+c-2$

or $a*d$, where $d=b+c-1$

$= a+d-1$

$= a+b+c-1-1$

Again, $(a*b)*c = (a+b-1)*c$

$= (a+b-1)+c-1$

$= a+b+c-2$

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