$\therefore a*(b*c)=(a*b)*c, \quad \forall a,b,c \in G.$

(III) Let 'e' be the identity elt. of G.

Then, $a*e = e*a = a$

Now, $a*e=a$

$\Rightarrow a+e-1=a$ (by defn).

$\Rightarrow e-1=0$

$\Rightarrow e=1.$

This shows that 1 is the identity elt. of G.

(IV) Let a_1 be the inverse of a.

Then, $a*a_1=e$.

$\Rightarrow a+a_1-1=1$

$\Rightarrow a+a_1=2$

$\Rightarrow a_1=2-a \in G.$

This shows that 2-a is the inverse of 'a'.

Hence $(G,*)$ is a group. #

03 Q:→ Let $\mathbb{Z}$ be the set of integers. We define $*$ on $\mathbb{Z}$ as follows:-

$a*b=a+b+1.$

Show that $(\mathbb{Z},*)$ is an infinite abelian group.

Solⁿ:→

Proof:- Given that

$\mathbb{Z}$ = the set of integers.

Define $*$ on $\mathbb{Z}$ be,

$a*b = a+b+1, \forall a,b \in \mathbb{Z}$

We now show that, $(\mathbb{Z},*)$ is an abelian group.

(I) Let $a,b \in \mathbb{Z}$

$\Rightarrow a+b \in \mathbb{Z}$

$\Rightarrow a+b+1 \in \mathbb{Z}$

$\Rightarrow a*b \in \mathbb{Z}, \forall a,b \in \mathbb{Z}$

(ii) Let, $a, b, c \in \mathbb{Z}$

Now,
$$a*(b*c) = a*(b+c+1) \text{ (by def}^n\text{)}$$
$$= a+(b+c+1)+1$$
$$= a+b+c+2.$$

Again,
$$(a*b)*c = (a+b+1)*c$$
$$=(a+b+1)+c+1$$
$$= a+b+c+2.$$

$$\therefore a*(b*c) = (a*b)*c \ , \forall a, b, c \in \mathbb{Z}$$

(iii) Let 'e' be the identity elt of $\mathbb{Z}$

~~$a*e = e*a = a$~~

$$a*e = a \ , \ \forall a \in \mathbb{Z}$$
$$\Rightarrow a+e+1 = a$$
$$\Rightarrow e+1 = 0$$
$$\Rightarrow e = -1$$

This shows that e is the identity elt of $\mathbb{Z}$.

(iv) Let ~~a be~~ $a \in \mathbb{Z}$ and a_1 be the inverse of 'a'.

$$a*a_1 = e$$
$$\Rightarrow a+a_1+1 = -1$$
$$\Rightarrow a+a_1 = -2$$
$$\Rightarrow a_1 = -2-a \in \mathbb{Z}$$

This shows that $-2-a$ is the inverse of 'a'.

(iv) Again we have,

$$a*b = a+b+1$$
$$= b+a+1$$
$$= b*a$$

$$\therefore a*b = b*a, \ \forall a, b \in \mathbb{Z}.$$

Hence $(\mathbb{Z}, *)$ is an ablian group.

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Since the set of integer $\mathbb{Z}$ is infinite therefore $(\mathbb{Z}, *)$ is an infinite ablian group.

Q:→ Prove that the set G consisting of the cube roots of unity is an abelian group under multiplication of complex numbers. Is G a cyclic group? Justify your answer.

$1^{\frac{1}{3}} = 1, \omega, \omega^2$

$\omega^3 = 1$
$\omega^4 = \omega$

Solⁿ:- Given that,

G = The set of cube roots of unity.
$= \{1, \omega, \omega^2\}$.

We now show that G is an abelian group w.r. to multiplication as binary operation.

(I) Clearly product of any two elt. of G is also an elt. of G.

(II) Again associative law hold good in G. being a subset of complex number.

(III) Clearly $1 \in G$, which is the identity elt. of G.

(IV) We have,

$$1^{-1} = 1 \in G$$
$$\omega^{-1} = \frac{1}{\omega} = \frac{\omega^2}{\omega^3} = \omega^2 \in G.$$
$$(\omega^2)^{-1} = \frac{1}{\omega^2} = \frac{\omega}{\omega^3} = \omega \in G.$$

This shows that every elt. of G has its multiplicative inverse.

(V) Again commutative law hold good in G. being a subset of the set of complex number.

Hence G is an ablian group.

2nd part:- G is a cyclic group generated by 'ω'

i.e, $G = \langle \omega \rangle$

For $1 \in G \Rightarrow 1 = \omega^3$
$\omega \in G \Rightarrow \omega = \omega^1$
$\omega^2 \in G \Rightarrow \omega^2 = \omega^2$.