$= a(\alpha,\beta) + a(\gamma,\delta)$

$= ax + ay$

$\therefore a(x+y) = ax + ay, \forall x, y \in V, \forall a \in \mathbb{R}$

V_3) Again we have—

$$(a+b)x = (a+b)(\alpha,\beta)$$
$$= ((a+b)\alpha, (a+b)\beta)$$
$$= (a\alpha + b\alpha, a\beta + b\beta)$$
$$= (a\alpha, a\beta) + (b\alpha, b\beta)$$
$$= a(\alpha,\beta) + b(\alpha,\beta)$$
$$= ax + bx$$

$\therefore (a+b)x = ax + bx, \forall x \in V$ & $\forall a, b \in \mathbb{R}$

V_4) Also we have —

$$(ab)x = (ab)(\alpha,\beta)$$
$$= ((ab)\alpha, (ab)\beta)$$
$$= (a(b\alpha), a(b\beta)) \text{ [By associativity in } \mathbb{R}]$$
$$= a(b\alpha, b\beta)$$
$$= a(b(\alpha,\beta))$$
$$= a.(bx)$$

$\therefore (ab)x = a(bx), \forall x \in V$ & $\forall a, b \in \mathbb{R}$

V_5) we have,

$$1.x = 1.(\alpha,\beta)$$
$$= (1.\alpha, 1.\beta)$$
$$= (\alpha,\beta)$$
$$= x$$

$\therefore 1.x = x, \forall x \in V$

where 1 is the multiplicative identity of the field $\mathbb{R}$.

Hence $V(\mathbb{R})$ is a vector space.//

Dt. 24.8.08

06.

Q. Prove that the set X of all real valued continuous functions defined on $[a,b]$ is a real vector space under

$$(f+g)(x) = f(x)+g(x)$$

$$(\alpha f)(x) = \alpha f(x),\ f,g \in X, \alpha \in \mathbb{R}.$$

Soln Given that, X is the set of all real valued continuous function defined on $[a,b]$ and define vector addition and scalar multiplication by:

$$(f+g)(x) = f(x)+g(x),\ \forall f,g \in X \rightarrow (i)$$

$$\& \ (\alpha f)(x) = \alpha f(x),\ \forall \alpha \in \mathbb{R}, f \in X \rightarrow (ii)$$

We now show that $X(\mathbb{R})$ is a vector space.

V_1) (i) Since sum of two continuous function is again a continuous function,

$$\therefore f+g \in X,\ \forall f,g \in X.$$

(ii) For any $f,g,h \in X$, we have –

$$\{f+(g+h)\}(x) = f(x)+(g+h)(x) \ [\text{using (i)}]$$

$$= f(x)+(g(x)+h(x))$$

$$= (f(x)+g(x))+h(x) \ [\text{using asso. in } \mathbb{R}]$$

$$= (f+g)x + h(x)$$

$$= \{(f+g)+h\}(x)$$

$$\therefore f+(g+h) = (f+g)+h,\ \forall f,g,h \in X.$$

(iii) Let us define $\hat{0}:[a,b] \rightarrow \mathbb{R}$ by,

$$\hat{0}(x) = 0,\ \forall x \in [a,b]$$

clearly, $\hat{0} \in X$.

We now have –

$$(f+\hat{0})(x) = f(x)+\hat{0}(x)$$

$$= f(x)+0$$

$$= f(x)$$

$$\therefore f+\hat{0} = f$$

S'lly, we can show that,

$$\hat{0}+f=f$$

$$\therefore f+\hat{0}=\hat{0}+f=f\ ,\ \forall f\in X$$

This shows that $\hat{0}$ is the additive identity of X.

(iv) For any $f\in X$, we define $-f:[a,b]\to\mathbb{R}$ by

$$(-f)(x)=-f(x).$$

We now have –

$$\{f+(-f)\}(x)=f(x)+(-f)(x)$$
$$=f(x)-f(x)$$
$$=0$$
$$=\hat{0}(x)$$

$$\therefore f+(-f)=\hat{0}$$

S'lly, we can show that

$$(-f)+f=\hat{0}.$$

$$\therefore f+(-f)=(-f)+f=\hat{0}$$

This shows that $-f$ is the additive inverse of f.

(v) We have,

$$(f+g)(x)=f(x)+g(x)$$
$$=g(x)+f(x)\quad[\text{by commu. in } \mathbb{R}]$$
$$=(g+f)x$$

$$\therefore f+g=g+f\ ,\ \forall f,g\in X.$$

Hence, X is an additive abelian group.

V_2) For any $\alpha\in\mathbb{R}$, we have –

$$\{\alpha(f+g)\}(x)=\alpha((f+g)(x))\quad[\text{using (ii)}]$$
$$=\alpha(f(x)+g(x))$$
$$=\alpha f(x)+\alpha g(x)$$
$$=(\alpha f)(x)+(\alpha g)(x)$$
$$=(\alpha f+\alpha g)(x)$$

$$\therefore \alpha(f+g)=(\alpha f+\alpha g)\ ,\ \forall \alpha\in\mathbb{R}\ \&\ \forall f,g\in X.$$